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THE SPECIFIC FEATURES OF THE LIMITING TRANSITION FROM A DISCRETE ELASTIC MEDIUM TO A CONTINUOUS ONE†

I. V. ANDRIANOV

Dnepropetrovsk

e-mail: igor_andrianov@hotmail.com

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The question of the link between the motions of a discrete elastic medium and its averaged approximation is discussed. It is shown that, in the case of a proper definition of solutions within the framework continuum theory, the motion of a discrete chain of masses can be sufficiently accurately described on the basis of these solutions. © 2002 Elsevier Science Ltd. All rights reserved.

1. THE INITIAL PROBLEM

The problem of motion of a chain of masses under the action of a force $f(0)$ applied to the end mass has been considered ([1]; see also [2–5]). The initial problem is described by the system of equations

$$\ddot{\delta}_j = \xi^2(\delta_{j+1} - 2\delta_j + \delta_{j-1}), \quad j = 1, 2, \dots, n; \quad \xi^2 = c/M \tag{1.1}$$

$$\delta_0 = -f(t), \quad \delta_{n+1} = 0 \tag{1.2}$$

with zero initial conditions. Here M is the mass of a particle, c is the stiffness of a spring and δ_j is the elastic force between the jth and the $(j-1)$th points.

In the case of large values of n, it is customary to use the continuous approximation of the discrete problem

$$\delta_{n}(x, t) = \xi^2 H^2 \delta_{xx}(x, t) \tag{1.3}$$

$$\delta(0, t) = -f(t), \quad \delta((n+1)H, t) = 0; \quad \delta(x, 0) = \delta_t(x, 0) = 0 \tag{1.4}$$

Here H is the distance between the point masses (the particles).

Having the solution of boundary-value problem (1.3), (1.4), the solution for a discrete medium can be recalculated using the formulae

$$\delta_i(t) = \delta(jH, t), \quad j = 1, 2, \dots, n \tag{1.5}$$

When $f(t) = -1$, the exact solution of boundary-value problem (1.3), (1.4) [6] is as follows:

$$\delta(x, t) = \theta\left(nH \arcsin\left|\sin\left(\frac{\pi}{2n}\xi t\right)\right| - x\right) \tag{1.6}$$

Here θ is the Heaviside function.

It therefore follows that, for all values of the time,

$$|\delta(x, t)| \leq 1 \tag{1.7}$$

We will now quote Kurchanov *et al.* [1, p. 990]: "Certain authors (for example [3, 6, 7]) have assumed that, when $f(t) = -1$, an inequality similar to (1.7) also holds for all components of the solution of system (1.1) in the case of sufficiently large n ... However, direct calculations for large values of n equal to 60, 80 and 120 have shown that the values of $\delta_j(t)$ at the discrete instants t_j^*, which are different for different

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j, are substantially greater (by tens of percent) than unity, which has direct practical consequences if one is talking, for example, about a rail structure". Kurchanov *et al.* [1] then carry out a rigorous mathematical investigation, the results of which are formulated in the form of a theorem and lead to the final conclusion [1, p. 993]: "In the language of mechanics, what has been stated above means that it is impossible to consider a one-dimensional, continuous medium during unlimited time intervals when analysing so-called local properties as the limiting case of a one-dimensional chain of point masses when there is an unlimited increase in the number of such points".

We shall call the phenomenon described (the appearance of bumps) the Kuchanov–Myshkis–Filimonov paradox.

2. VIBRATION FREQUENCIES

Putting $f(t) = 0$, we will investigate the relation between the natural vibration frequencies of discrete system (1.1), (1.2) and continuous system (1.3), (1.4).

As a criterion of the sufficient accuracy of the continuous approximation, we shall use the closeness of the frequencies of the discrete and continuous systems. Of course, this criterion is extremely arbitrary but, since the required solution within the framework of the continuous model is considered as an expansion in the characteristic forms of the vibrations, an increase in the accuracy of the determination of the corresponding frequencies undoubtedly increases the accuracy of the solution of the problem of the natural and forced vibrations of the chain.

Discrete system (1.1), (1.2) has $n + 1$ natural frequencies which are described by the formulae

$$\omega_k = 2\xi \sin\frac{k\pi}{2(n+1)}, \quad k = 1, 2, \ldots, n+1 \tag{2.1}$$

Continuous system (1.3), (1.4) has a discrete infinite spectrum

$$\alpha_k = \pi\xi\frac{k}{n+1}, \quad k = 1, 2, \ldots \tag{2.2}$$

Expressions (2.2) approximate the vibration frequencies of discrete system (2.1). For the first frequencies, the approximation is good but, for large values of k, it is bad and, let us say, ω_{n+1} is determined with an error of 50% (a numerical coefficient π instead of 2). The accuracy of approximation (2.2) can be increased (more about this later) but, for the present, we note the following: the frequencies of the continuous system ω_{n+2}, ω_{n+3}, etc. are unrelated to discrete system (1.1), (1.2). These are parasitic frequencies and if one is discussing the investigation of discrete system (1.1), (1.2), they must not be taken into account.

Note that no problems arise in the given case when a one-dimensional system is considered: it is necessary to approximate the system with n frequencies and, for this purpose, we use the first n frequencies of the continuous system. Possibly, a more detailed investigation is required in the two- or three-dimensional case.

3. THE MOTION OF A CHAIN OF MASSES UNDER THE ACTION OF A FORCE

We will now consider boundary-value problem (1.3)–(1.5). Putting $f(t) = -1$, we shall seek the solution in the form

$$\delta = -1 + \frac{x}{(n+1)H} + u(x, t) \tag{3.1}$$

Then, the function $u(x, t)$ is defined by the boundary-value problem

$$\frac{\partial^2 u}{\partial t^2} = \xi^2 H^2 \frac{\partial^2 u}{\partial x^2} \tag{3.2}$$

$$u(0, t) = u((n+1)H, t) = 0; \quad u(x, 0) = 1 - \frac{x}{(n+1)H}, \quad u_t(x, 0) = 0 \tag{3.3}$$

The solution of boundary-value problem (3.2), (3.3) is found using Fourier's method

$$u(x,t) = S_\infty, \quad S_m = \frac{(n+1)}{\pi}\sum_{k=1}^{m}\frac{1}{k}\sin\frac{k\pi x}{(n+1)H}\cos(\alpha_k t) \tag{3.4}$$

The quantities α_k are defined by expression (2.2).

Solution (3.1), (3.4) describes the longitudinal motion of a rod. If the motion of a chain of particles (1.1), (1.2) is approximated, then it is only necessary to retain $n + 1$ harmonics in the infinite sum; the remaining harmonics have no relation to the motion of the chain of masses. In other words, the motion of discrete system (1.1) can be approximately described by the expression

$$\delta = -1 + \frac{x}{(n+1)H} + S_{n+1} \tag{3.5}$$

Numerical calculations using expression (3.5) when $n = 60$, 80 and 120 show that the "bumps" discovered earlier in [1, 2] actually occur and that δ can be greater than unity.

Psychologically, of course, the existence of exact solution (1.6) wins over but it should be borne in mind that the continuous approximation must only serve to determine those vibration frequencies and forms of motion of the discrete chain which are characteristic of the chain.

4. INCREASING THE ACCURACY OF THE CONTINUOUS APPROXIMATION

Solution (3.5) describes the motion of a chain of masses qualitatively correctly. However, quantitatively, the accuracy is low, since the vibration forms, which are close to the $(n + 1)$th vibration are poorly described by approximation (1.3).

The following approach can be proposed for refining the solution.

We rewrite system (1.1) using a difference operator in the form [8–10]

$$\frac{\partial^2\delta}{\partial t^2} + 4\xi^2\sin^2\left(-\frac{iH}{2}\frac{\partial}{\partial x}\right)\delta = 0 \tag{4.1}$$

Expansion of the difference operator in a Taylor series in the neighbourhood of zero gives

$$\sin^2\left(-\frac{iH}{2}\frac{\partial}{\partial x}\right)\delta = \frac{1}{4}H^2\left[\frac{\partial^2\delta}{\partial x^2} + \frac{H^2}{12}\frac{\partial^4\delta}{\partial x^4} + \frac{H^4}{360}\frac{\partial^6\delta}{\partial x^6} + \dots\right] \tag{4.2}$$

On retaining just the first term in expansion (4.2), we obtain the usual continuous approximation (1.3). On retaining three terms, we obtain the higher-order approximation

$$\frac{\partial^2\delta}{\partial t^2} = \xi^2 H^2\left[\frac{\partial^2\delta}{\partial x^2} + \frac{H^2}{12}\frac{\partial^4\delta}{\partial x^4} + \frac{H^4}{360}\frac{\partial^6\delta}{\partial x^6}\right] \tag{4.3}$$

Comparison of the $(n + 1)$th frequency for the continuous chain (4.3) with the $(n + 1)$th frequency of the discrete system shows a substantial increase in the accuracy (a numerical coefficient of 2.11 instead 2 in the exact solution). Hence, to describe the motion of a chain of masses, it is best to use solution (3.5), putting

$$\alpha_k = \pi\xi\frac{k}{n+1}\left[1 - \frac{\pi^2k^2}{12(n+1)^2} + \frac{\pi^4k^4}{360(n+1)^4}\right]^{1/2}$$

Note that the use of Taylor's formula to replace the difference operator has been known for a long time. Distrust in this method, because of its particularly formal character, has been expressed by Mandel'shtam [11] in his lectures on the oscillation theory. He obtained evidence for this, for example, within the framework of moment theories of elasticity (the theory of media with long-range action) [12] or the method of differential approximation [13].

Numerical calculations using Eq. (4.3) confirm the existence of the bumps discovered earlier in [1, 2].

5. "HIGH-FREQUENCY AVERAGING" AND COMPOSITE EQUATIONS

The difference operator in Eq. (4.1) can also be considered in the neighbourhood of the identity transformation

$$\sin^2\left(-\frac{iH}{2}\frac{\partial}{\partial x}\right)=1+\frac{H^2\partial^2}{4\partial x^2}+\ldots \tag{5.1}$$

The continuous approximation can then be written as (see [14–16])

$$\left(\frac{\partial^2}{\partial t^2}+4\xi^2+\xi^2H^2\frac{\partial^2}{\partial x^2}\right)\delta=0 \tag{5.2}$$

Equation (5.2) describes the vibrations of the chain in a mode which is close to a "saw tooth" mode ($\delta_i=-\delta_{i-1}$).

The existence of the continuous approximations (1.3) and (5.2) enables one to construct a composite equation [8–10, 17], which is roughly adequate over the whole range of frequencies and vibration modes of a chain of masses. The basic idea behind the method of composite equations can be formulated as follows [17]:

a) the terms of the differential equations, which, when neglected, generate inhomogeneities, are determined;

b) these terms are approximated as simply as possible while retaining their essential features in the domain of inhomogeneity.

We will now construct a composite equation such that, when $k \ll n+1$, it is approximately identical to Eq. (1.3) and, when $k=n+1$, the exact value of the frequency is obtained. As a result, we have

$$\left(1-\alpha^2H^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2\delta}{\partial t^2}-\xi^2H^2\frac{\partial^2\delta}{\partial x^2}=0,\quad \alpha^2=\frac{1}{4}-\frac{1}{\pi^2} \tag{5.3}$$

The expression for the kth vibration frequency

$$\alpha_k=\pi\xi\frac{k}{\sqrt{(n+1)^2+\pi^2\alpha^2k^2}} \tag{5.4}$$

gives values close to the frequencies of a discrete chain for all k from 1 to $n+1$. The greatest error occurs when $k/(n+1)=1/2$. The numerical factor in the exact value of the frequency is equal to $\sqrt{2}$ while, by formula (5.4), we have 1.34. It is therefore logical to use Eq. (5.3) to calculate the motion of a system of masses. The solution in this case has the form of (3.5) with α_k in the form of (5.4). Numerical calculations using solution (3.5), (5.4) confirm the existence of the bumps discovered earlier in [1, 2].

6. CONCLUSION

Ulam [18, pp. 89, 90] writes: "The usual introduction of the continuum leaves much to be discussed and examined critically ... The finite system of [N] ordinary differential equations 'becomes' in the limit $N=\infty$ one or several *partial* differential equations. The Newtonian laws of conservation of energy and momentum are seemingly correctly formulated for the limiting case of a continuum. There appears at once, however, at least one possible objection to the unrestricted validity of this formulation. For the very fact that the limiting equations imply tacitly the continuity and differentiability of the functions describing the motion seems to impose various *constraints* on the possible motions of the approximating finite systems. Indeed, at any stage of the limiting process, it is quite conceivable for two neighbouring particles to be moving in opposite directions with a relative velocity which need not tend to zero as N becomes infinite, whereas the continuity imposed on the solution of the limiting continuum excludes such a situation ... In some cases, therefore, the usual differential equations of hydrodynamics may constitute a misleading description of the physical process".

The fact about which Ulam has written on the basis of general considerations [18] has been discovered numerically and investigated mathematically [1–5]. Strictly speaking there is nothing unexpected in the fact that the averaged relations do not enable one to obtain the true results for the whole spectrum of effects. The question can be formulated in the following manner: is it possible to construct a partial differential equation which describes the motion of a discrete system sufficiently well? The method of composite equations is one of the possible routes in the search for an answer to this question.

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